1.3 – Matrices and Matrix Operations

Definitions: A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix. The **size** of a matrix that has m rows and n columns is $m \times n$ (read "m by n").

#**6** Use the following matrices to compute the indicated expression if it is defined.

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

a. $\left(2D^T - E\right) A$
b. $(4B)C + 2B$
c. $(-AC)^T + 5D^T$
d. $\left(BA^T - 2C\right)^T$
e. $B^T(CC^T - A^TA)$
f. $D^TE^T - (ED)^T$

Definition: If A is any $m \times n$ matrix, then the **transpose of A**, denoted by A^T, is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.

#17 Use the column-row expansion of AB to express this product as a sum of matrix products.

$A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & 1 \end{bmatrix}$

Definitions: The **column-row expansion** of AB is $AB = \mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \ldots + \mathbf{c}_r\mathbf{r}_r$, where $\mathbf{c}_i$ are column vectors of A and $\mathbf{r}_i$ are row vectors of B.

The linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

Can be expressed using matrix multiplication.

Definition: When a linear system is written using a matrix product

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or $A\mathbf{x} = \mathbf{b}$, A is called the **coefficient matrix** of the system.

#**13** a. Express the matrix equation as a system of linear equations.

$$\begin{bmatrix} 5 & 6 & -7 \\ -1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

Definition: Two matrices are defined to be **equal** if they have the same size and their corresponding entries are equal.

Definition: If $A_1,\ A_2,\ \ldots,\ A_r$ are matrices of the same size, and if $c_1,\ c_2,\ \ldots,\ c_r$ are scalars, then an expression of the form $c_1A_1 + c_2A_2 + \ldots + c_rA_r$ is called a **linear combination** of $A_1,\ A_2,\ \ldots,\ A_r$ with **coefficients** $c_1,\ c_2,\ \ldots,\ c_r$.

#22 Example 6 of section 1.2 presents the linear system

$$\begin{array}{rrrrrrl} x_1 & +3x_2 & -2x_3 & & +2x_5 & & = 0 \\ 2x_1 & +6x_2 & -5x_3 & -2x_4 & +4x_5 & -3x_6 & = 0 \\ & & 5x_3 & +10x_4 & & +15x_6 & = 0 \\ 2x_1 & +6x_2 & & +8x_4 & +4x_5 & +18x_6 & = 0 \end{array}$$

and the reduced row echelon form of its associated augmented matrix as

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Express the solution as a linear combination of column vectors that contain only numerical entries.

Theorem 1.3.1 If A is an $m \times n$ matrix, and if $\mathbf{x}$ is an $n \times 1$ column vector, then the product $A\mathbf{x}$ can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of $\mathbf{x}$.

Definition: The **main diagonal** of a square matrix contains entries from the upper left to lower right corners.

Definition: If A is a square matrix, then the **trace of A**, denoted by $tr\,(A)$, is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.